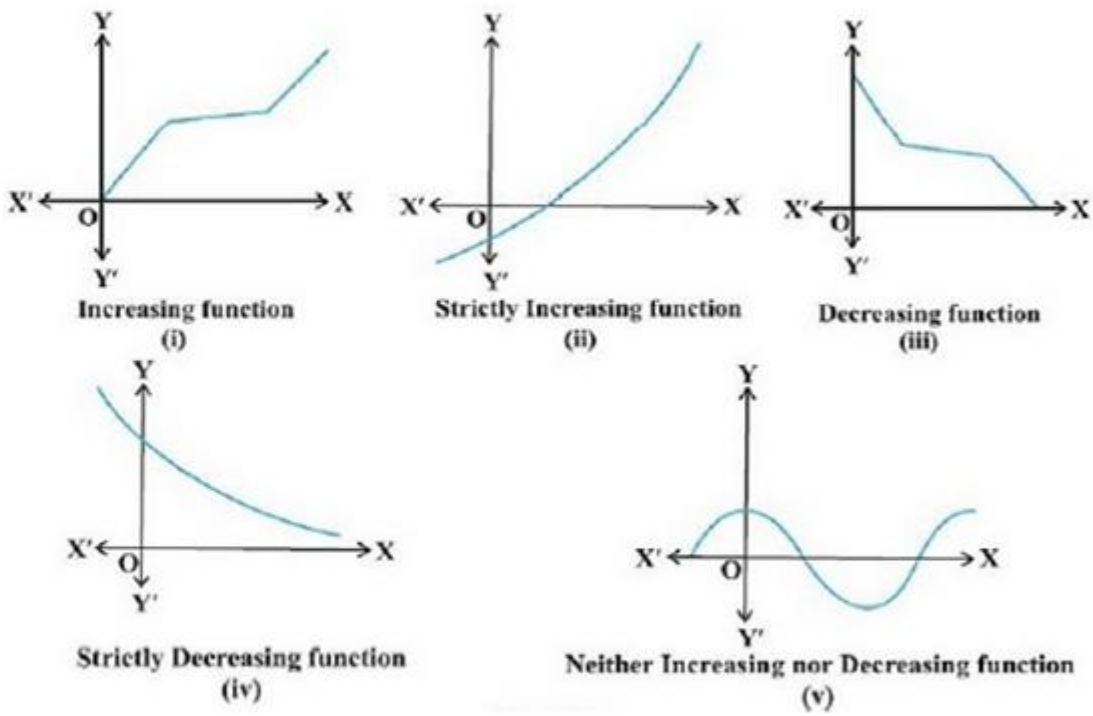
L- 6 Applications Of Derivatives (Handout Module-1)

Increasing and Decreasing Functions

Let f be a real valued function defined in an interval D (a subset of R), then f is called an increasing function in an interval D_1 (a subset of D) iff for all $x_1, x_2 \in D_1,\ x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ and
f is called a strictly increasing function in D_1 iff for all
$x_1, x_2 \in D_1,\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Analogously, f is called a decreasing function in an interval D_2 iff for all
$x_1, x_2 \in D_2,\ x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ and
f is called a strictly decreasing function in D_2 iff for all
$x_1, x_2 \in D_2,\ x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

For graphical representation of increasing and decreasing functions see figure shown below:



Definition:- Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, strictly increasing, decreasing or strictly decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, strictly increasing, decreasing or strictly decreasing, respectively, in I.

A function f is said to be increasing at x_0 if there exists an interval $I = (x_0 - h, x_0 + h), h > 0$ such that for $x_1, x_2 \in I$.

$$x_1 < x_2 \; in \; I \Rightarrow f(x_1) \leq f(x_2)$$

Similarly, the other cases can be clarified.

Theorem 1. If a function f is continuous in $[a, b]$, and derivable in (a, b) and

i) $f^/(x) \geq 0 \; for \; all \; x \in (a, b)$, then f is increasing in $[a, b]$.

ii) $f^/(x) > 0 \; for \; all \; x \in (a, b)$, then f is strictly increasing in $[a, b]$.

Theorem 2. If a function f is continuous in $[a, b]$, and derivable in (a, b) and

$i)$ $f^/(x) \leq 0 \; for \; all \; x \in (a, b)$, then f is decreasing in $[a, b]$.

$ii)$ $f^/(x) < 0 \; for \; all \; x \in (a, b)$, then f is strictly decreasing in $[a, b]$.

If $f^/(x) = 0 \; for \; all \; x \in (a, b)$, then f is a constant function in $[a, b]$.

Remark:- The formal proofs of these theorems are based on Lagrange's Mean Value Theorem.

Corollary:- If a function f is continuous in $[a, b]$, and derivable in $(a, b)$and

i) $f^/(x) > 0$ for all x in (a, b) except for a finite number of points where $f^/(x) = 0$, then $f(x)$ is strictly increasing in $[a, b]$.

ii) $f^/(x) < 0$ for all x in (a, b) except for a finite number of points where $f^/(x) = 0$, then $f(x)$ is strictly decreasing in $[a, b]$.

Example1. Prove that the function $f(x) = x^3 - 6x^2 + 15x - 18$ is strictly increasing on R.

Solution:- Given $f(x) = x^3 - 6x^2 + 15x - 18, D_f = R$

Differentiating it with respect to x, we get

$$f^{/}(x) = 3x^2 - 12x + 15 = 3(x^2 - 4x + 5)$$

$$= 3[(x-2)^2 + 1] \geq 3$$

$$\Rightarrow f^{/}(x) > 0 \; for \; all \; x \in R$$

$\Rightarrow f(x)$ is strictly increasing function for all $x \in R$.

Example:-2. Show that the function $f(x) = x^2 - 3x + 1$ is neither increasing nor decreasing on $(0,3)$.

Solution:- Given $f(x) = x^2 - 3x + 1 \Rightarrow f'(x) = 2x - 3$.

Now$f'(x) > 0$ when$2x - 3 > 0$ $i.e \; when \; x > \frac{3}{2}$

$\Rightarrow$ the given function is increasing in $\left[\frac{3}{2}, \bowtie\right)$.

And $f'(x) < 0$ when$2x - 3 < 0$ $i.e \; when \; x < \frac{3}{2}$

$\Rightarrow$ the given function is decreasing in $\left(-\bowtie, \frac{3}{2}\right]$.

Hence, in particular, the given function is increasing in $\left[\frac{3}{2}, 3\right)$ and decreasing in $\left(0, \frac{3}{2}\right]$
Therefore, it is neither increasing nor decreasing on $(0,3)$.

Example 3. Find the intervals in which the function f given by

$$f(x) = \frac{4\sin x}{2+\cos x} - x, \quad 0 \le x \le 2\pi, is$$

i) Strictly increasing ii) strictly decreasing

Solution:- Given $f(x) = \frac{4\sin x}{2+\cos x} - x, \;\; 0 \le x \le 2\pi$

It is differentiable for all $x \in [0,2\pi]$.

$$f^{/}(x) = \frac{(2+\cos x)4\cos x - 4\sin x\,(-\sin x)}{(2+\cos x)^2} - 1 = \frac{8\cos x + 4(cos^2 x + sin^2 x) - (2+\cos x)^2}{(2+\cos x)^2}$$

$$f^{/}(x) = \frac{8\cos x + 4 - 4 - cos^2 x - 4\cos x}{(2+\cos x)^2} = \frac{4\cos x - cos^2 x}{(2+\cos x)^2} = \frac{(4-\cos x)\cos x}{(2+\cos x)^2}$$

$$f^{/}(x) = \frac{(4-\cos x)\cos x}{(2+\cos x)^2}$$

We note that $-1 \le \cos x \le 1, \;\; for\, all\, x \in [0,2\pi]$

$$\Rightarrow 1 \ge -\cos x \ge -1, \;\; for\, all\, x \in [0,2\pi]$$

$$\Rightarrow 5 \ge 4 - \cos x \ge 3, \;\; for\, all\, x \in [0,2\pi]$$

$$\Rightarrow 4 - \cos x > 0, also\; (2 + \cos x)^2 > 0, for\, all\, x \in [0,2\pi]$$

$$\therefore \frac{(4 - \cos x)}{(2 + \cos x)^2} > 0\;, for\, all\, x \in [0,2\pi]$$

i) f is strictly increasing iff $f^{/}(x) > 0$

$\Rightarrow \frac{(4-\cos x)\cos x}{(2+\cos x)^2} > 0 \Rightarrow \cos x > 0$

$\Rightarrow x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

$\therefore$ f is strictly increasing in $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$.

ii) f is strictly decreasing iff $f^{/}(x) < 0$

$\Rightarrow \frac{(4-\cos x)\cos x}{(2+\cos x)^2} < 0 \Rightarrow \cos x < 0$

$\Rightarrow x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$\therefore$ f is strictly decreasing in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.